Квантова механика упражнения

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I. АЛГЕБРА НА ЛИНЕЙНИ ЕРМИТОВИ ОПЕРАТОРИ

Оператор

Правило, което на обект от дадено пространство съпоставя обект от същото или друго пространство се нарича оператор.

Примери за оператори

Диференциране

$$\frac{df(x)}{dx} \to g(x)$$

$$\frac{dx^2}{dx} = 2x$$

Интегриране

$$\int f(x)\,dx \to g(x)$$

$$\int x^3 dx = \frac{1}{4}x^4$$

Отражение

$$\hat{I} f(x) = f(-x)$$

$$\hat{I}\left(x^2 + x\right) = x^2 - x$$

Транслация

$$\hat{T}_a f(x) = f(x+a)$$

$$\hat{T}_a\left(x^2 + x\right) = (x+a)^2 + x + a$$

Комплексно спрягане

$$\hat{K} f(x) = f^*(x)$$

$$\hat{K}\left(3x - ix^2\right) = 3x + ix^2$$

Линейни оператори

$$\hat{A}\sum\lambda_k f_k = \sum\lambda_k \hat{A} f_k$$

$$\left(f\left(x\right)+g\left(x\right)\right)_x' = \left(f\left(x\right)\right)_x' + \left(g\left(x\right)\right)_x'$$

$$\int\left(f\left(x\right)+g\left(x\right)\right)dx = \int f\left(x\right)dx + \int g\left(x\right)dx$$

Умножение на два и по-вече оператора

$$\hat{A}\hat{B}f = \hat{A}\left(\hat{B}f\right)$$

$$\hat{A}\hat{B}\hat{C}f = \hat{A}\left(\hat{B}\left(\hat{C}f\right)\right)$$

Не комутационно свойство на операторите

$$\hat{D}_x f = \frac{d}{dx}f$$

$$\hat{I}f\left(x\right) = f\left(-x\right)$$

$$\begin{aligned}\hat{I}\hat{D}_x\left(x^4+x^3\right) &= \hat{I}\left(4x^3+3x^2\right) = -4x^3+3x^2\\ \hat{D}_x\hat{I}\left(x^4+x^3\right) &= \hat{D}_x\left(x^4-x^3\right) = 4x^3-3x^2\end{aligned}$$

$\hat{D}_x$ и $\hat{I}$ не комутират ($\hat{I}\hat{D}_x \neq \hat{D}_x\hat{I}$)
Комутатор и антикомутатор

$$\left[\hat{A},\hat{B}\right] = \hat{A}\hat{B} - \hat{B}\hat{A}$$

ако $\left[\hat{A},\hat{B}\right] = 0$ то $\hat{A}$ и $\hat{B}$ комутират ($\hat{A}\hat{B} = \hat{B}\hat{A}$)

$$\left\{\hat{A},\hat{B}\right\} = \hat{A}\hat{B} + \hat{B}\hat{A}$$

ако $\left\{\hat{A},\hat{B}\right\} = 0$ то $\hat{A}$ и $\hat{B}$ антикомутират ($\hat{A}\hat{B} = -\hat{B}\hat{A}$)

Задачи:

Ако $\hat{A}$ и $\hat{B}$ са линейни оператори който не комутират докажете че:

$$\left[\hat{A},\hat{B}\right] = -\left[\hat{B},\hat{A}\right]$$

$$\left\{\hat{A},\hat{B}\right\} = \left\{\hat{B},\hat{A}\right\}$$

Докажете тъждествата:

$$\left[\hat{A}\hat{B},\hat{C}\right] = \hat{A}\left[\hat{B},\hat{C}\right] + \left[\hat{A},\hat{C}\right]\hat{B}$$

$$\left[\hat{A},\hat{B}\hat{C}\right] = \left[\hat{A},\hat{B}\right]\hat{C} + \hat{B}\left[\hat{A},\hat{C}\right]$$

$$\left\{\hat{A},\hat{B}\hat{C}\right\} = \left[\hat{A},\hat{B}\right]\hat{C} + \hat{B}\left\{\hat{A},\hat{C}\right\}$$

Докажете тъждеството на Якоби:

$$\left[\hat{A},\left[\hat{B},\hat{C}\right]\right] + \left[\hat{B},\left[\hat{C},\hat{A}\right]\right] + \left[\hat{C},\left[\hat{A},\hat{B}\right]\right] = 0$$

Ако $\hat{A}$ и $\hat{B}$ са линейни оператори който комутират($\hat{A}\hat{B} = \hat{B}\hat{A}$) докажете:

$$\left[\hat{A},\lambda\hat{B}\right] = 0$$

$$\left[\hat{A},\hat{B}^n\right] = 0$$

$$\left[\hat{A}^l,\hat{B}^n\right] = 0$$

$$\left[\hat{A},P_n\left(\hat{B}\right)\right] = 0$$

$$\left[Q_m\left(\hat{A}\right),P_n\left(\hat{B}\right)\right] = 0$$

$$\left[f\left(\hat{A}\right),g\left(\hat{B}\right)\right] = 0$$

тук λ е реално число, n и l са цели числа а P_n и Q_m са полиноми съответно от степен n и m.

Ако $\hat{A}$ и $\hat{B}$ са линейни оператори който комутират($\hat{A}\hat{B} = \hat{B}\hat{A}$) намерете $\left[\hat{A},\exp\left(\hat{B}\right)\right]$:

Ако $\hat{A}$ и $\hat{B}$ са линейни оператори да се докаже:

$$\exp\left(\alpha\hat{A}\right)\hat{B}\exp\left(-\alpha\hat{A}\right) = \hat{B} + \alpha\left[\hat{A},\hat{B}\right] + \frac{\alpha^2}{2!}\left[\hat{A},\left[\hat{A},\hat{B}\right]\right] + \frac{\alpha^3}{3!}\left[\hat{A},\left[\hat{A},\left[\hat{A},\hat{B}\right]\right]\right] + ...$$

$$\exp\left(-\alpha\hat{A}\right)\hat{B}\exp\left(\alpha\hat{A}\right) = \hat{B} + \alpha\left[\hat{B},\hat{A}\right] + \frac{\alpha^2}{2!}\left[\left[\hat{B},\hat{A}\right],\hat{A}\right] + \frac{\alpha^3}{3!}\left[\left[\left[\hat{B},\hat{A}\right],\hat{A}\right],\hat{A}\right] + ...$$